Department of Mathematics

Temple University

COMPARISONS AMONG ESTIMATORS OF A SCALE PARAMETER OF THE BETA-STACY DISTRIBUTION

by N. L. Johnson *, University of North Carolina at Chapel Hill

and Samuel Kotz**, Temple University, Philadelphia

April, 1970

* Supported by Air Force Office of Scientific Research under [redacted] AFOSR-68-1415.

** Supported by Air Force Office of Scientific Research under [redacted] AFOSR-68-1411.

14

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Temple University Department of Mathematics Philadelphia, Pennsylvania 19122	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

COMPARISONS AMONG ESTIMATORS OF A SCALE PARAMETER OF THE BETA-STACY DISTRIBUTION

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Scientific Interim

5. AUTHOR(S) (First name, middle initial, last name)

N. L. Johnson and Samuel Kotz

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
April 1970	12	0

8a. CONTRACT OR GRANT NO. AFOSR-68-1411 b. PROJECT NO. 9769 c. 6 02F d. 304	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) AFOSR - TR - 70 - 0796

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES TECH, OTHER	12. SPONSORING MILITARY ACTIVITY Air Force Office of Scientific Research 1400 Wilson Boulevard (NM) Arlington, Virginia 22209

13. ABSTRACT

A comparative survey is made of several estimators of a scale parameter of the Beta-Stacy distribution. In essence, some optimum technique is sought to permit more insight into the statistics of a joint distribution of two random variables when one set has a generalized gamma distribution. Comparisons are made on the accuracy lost when no additional statistics are available on the set having the generalized gamma distribution. This work is a logical extension of the bivariate warning-time/failure time distribution effort conducted by Mihran and Hultquist (1967).

DD FORM 1473

Comparisons among Estimators of a Scale Parameter of the Beta-Stacy Distribution

By N.L. Johnson*, University of North Carolina at Chapel Hill

and S. Kotz**, Temple University, Philadelphia

The Beta-Stacy distribution, as defined by Mihram and Hultquist [1] is the joint distribution of two random variables, X_1 and X_2, when (i) X_1 has a Stacy (generalized gamma) distribution with density function

$$(1)\quad p_{X_1}(x_1) = [\Gamma(\alpha)]^{-1} c a^{-\alpha c} x_1^{\alpha c-1} \exp[-(x_1/a)^c]$$

$$(x_1 > 0;\ \alpha, c, a > 0)$$

and

(ii) the conditional distribution of X_2, given X_1, is beta with parameters θ_1, θ_2 and range 0 to X_1, so that

$$(2)\quad p_{X_1|X_2}(x_1|x_2) = [B(\theta_1,\theta_2)]^{-1}(x_2/x_1)^{\theta_1-1}(1-x_2/x_1)^{\theta_2-1}x_1^{-1}$$

$$(0 < x_2 < x_1;\ \theta_1, \theta_2 > 0).$$

---

* Supported by Air Force Office of Scientific Research under contract AFOSR-68-1415.

** Supported by Air Force Office of Scientific Research under contract AFOSR-68-[illegible].

Given n independent pairs of random variables $\{ X_{1j}, X_{2j} \}$

$(j = 1,2,\ldots,n)$, each having the Beta-Stacy distribution is defin

It is clear that, given $\{ X_{1j} \}$, the observations $\{X_{2j}\}$ on the second variable $\{ X_{2j} \}$ provide no further information on the values of the parameters a,c and α of the common distribution of the X_{1j}'s.

In particular, if c and α are known then $\sum_{j=1}^{n} X_{1j}^{c}$ is a sufficient statistics for a, and the minimum variance unbiased estimator of a is

(3) $$\hat{a}_1 = \frac{\Gamma(n\alpha)}{\Gamma(n\alpha + c^{-1})} \left(\sum_{j=1}^{n} X_{1j}^{c}\right)^{1/c} .$$

Its variance is

(4) $$\operatorname{var}(\hat{a}_1) = a^2 \left\{ \frac{\Gamma(n\alpha)\Gamma(n\alpha + 2c^{-1})}{[\Gamma(n\alpha + c^{-1})]^2} - 1 \right\}$$

while the Cramér-Rao lower bound for unbiased estimators of a is

(5) $$a^2(n\alpha c^2)^{-1} .$$

The ratio of $\operatorname{var}(\hat{a}_1)$ to $a^2(n\alpha c^2)^{-1}$ tends to 1 as n tends to infinity.

Mihram and Hultquist [1] studied the problem of estimating a when observations are available on $\{X_{2j}\}$, but not on $\{X_{1j}\}$, it being supposed that θ_1 and θ_2 are known, as well as c and α. They suggested two estimators, based on the geometric and arithmetic mean of the X_{2j}'s, respectively:

$$(6.1) \quad \hat{a}_2 = \left[\frac{\Gamma(\theta_1)\,\Gamma(\alpha)\,\Gamma(\theta_1+\theta_2+n^{-1})}{\Gamma(\theta_1+n^{-1})\Gamma(\alpha+(nc)^{-1})\Gamma(\theta_1+\theta_2)} \right]^n \prod_{j=1}^{n} X_{2j}^{1/n}$$

$$(6.2) \quad \hat{a}_3 = \frac{(\theta_1+\theta_2)\,\Gamma(\alpha)}{\theta_1\,\Gamma(\alpha+c^{-1})} \cdot \frac{1}{n} \sum_{j=1}^{n} X_{2j} \ .$$

Since $$E[X_2^{\,s}|X_1] = X_1^{\,s}\,\frac{\Gamma(\theta_1+s)\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1+\theta_2+s)\,\Gamma(\theta_1)}$$

and $E[X_1^{\,t}] = a^t\,\Gamma(\alpha+sc^{-1})/\Gamma(\alpha)$,

it follows that

$$(7) \quad E[X_1^{\,t}X_2^{\,s}] = a^{s+t}\,\frac{\Gamma(\alpha+(s+t)c^{-1})}{\Gamma(\alpha)} \cdot \frac{\Gamma(\theta_1+s)\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1+\theta_2+s)\Gamma(\theta_1)} \ .$$

From (7) we obtain

$$(8.1) \quad \mathrm{var}(\hat{a}_2) = a^2 \left\{ \left[\frac{\Gamma(\alpha)\,\Gamma(\theta_1)\Gamma(\alpha+2(nc)^{-1})\Gamma(\theta_1+2n^{-1})\left\{\Gamma(\theta_1+\theta_2+2n^{-1})\right\}^2}{\Gamma(\theta_1+\theta_2)\Gamma(\theta_1+\theta_2+2n^{-1})\left\{\Gamma(\alpha+(nc)^{-1})\Gamma(\theta_1+n^{-1})\right\}^2} \right]^n - 1 \right\},$$

$$(8.2)\quad \mathrm{var}(\hat{a}_3) = a^2 n^{-1}\left\{\frac{(\theta_1+1)(\theta_1+\theta_2)}{\theta_1(\theta_1+\theta_2+1)} \cdot \frac{\Gamma(\alpha+2c^{-1})\Gamma(\alpha)}{[\Gamma(\alpha+c^{-1})]^2} - 1\right\}$$

$$(8.3)\quad \mathrm{Cov}(\hat{a}_2,\hat{a}_3) = a^2\left\{\frac{1+(n\theta_1)^{-1}}{1+[n(\theta_1+\theta_2)]^{-1}} \cdot \frac{\Gamma(\alpha)\,\Gamma(\alpha+(n+1)(nc)^{-1})}{\Gamma(\alpha+c^{-1})\,\Gamma(\alpha+(nc)^{-1})} - 1\right\}.$$

(Formulae (8.1) and (8.3) disagree with the corresponding formulae in [1]).

As n tends to infinity, we have

$$(9.1)\quad a^{-2}\lim_{n\to\infty} n\,\mathrm{var}(a_2) = c^{-2}\psi'(\alpha) + \psi'(\theta_1) - \psi'(\theta_1+\theta_2)$$

$$(9.3)\quad a^{-2}\lim_{n\to\infty} \mathrm{cov}(\hat{a}_2,\hat{a}_3) = c^{-1}[\psi(\alpha+c^{-1}) - \psi(\alpha)] + \theta_2\theta_1^{-1}(\theta_1+\theta_2)^{-1},$$

where $\psi(y) = \frac{d}{dy}(\log\Gamma(y))$ and $\psi'(y) = \frac{d}{dy}(\psi(y))$.

And, of course, for all n

$$(9.2)\quad a^{-2}\,n\,\mathrm{var}\,(a_3) = \frac{(\theta_1+1)(\theta_1+\theta_2)}{\theta_1(\theta_1+\theta_2+1)} \cdot \frac{\Gamma(\alpha+2c^{-1})\Gamma(\alpha)}{\{\Gamma(\alpha+c^{-1})\}^2} - 1.$$

Table 1 gives values of:

$a^{-2}\,n\,\mathrm{var}(\hat{a}_2)$, $a^{-2}\,n\,\mathrm{var}(\hat{a}_3)$, and $\mathrm{corr}(\hat{a}_2,\hat{a}_3)$ for selected values of $c,\alpha,\theta_1,\theta_2$ and n. The values for $n = \infty$ are calculated from the right hand sides of (9.1) - (9.3). For each set of values of c,α,θ_1 and θ_2, n is taken equal to 100. In a few cases values are given also for $n = 10, 20, 50$ and ∞. These should suffice to indicate the variation with n, which is not marked.

In order to see how much accuracy has been lost by ignorance of $\{X_{1j}\}$ the values of $a^{-2}\,n\,\mathrm{var}\,(\hat{a}_j)$ $(j = 1,2)$ can be compared with the corresponding value

$$a^{-2}\,n\,\mathrm{var}\,(\hat{a}_1) = n\left\{\frac{\Gamma(n\alpha)\Gamma\,(n\alpha+2c^{-1})}{[\Gamma(n\alpha+c^{-1})]^2} - 1\right\}$$

for the minimum variance unbiased estimator $\hat{a}_1$. These values are given in Table 2.

Using the approximation $\psi(\alpha) \doteqdot \log(\alpha-\frac{1}{2})$ we see, from (9.1), that

$$(10)\qquad a^{-2}\lim_{n\to\infty} n\,\mathrm{var}(\hat{a}_2) \doteqdot c^{-2}(\alpha-\tfrac{1}{2})^{-1} + \psi'(\theta_1) - \psi'(\theta_1+\theta_2)$$

while

$$(11)\qquad a^{-2}\lim_{n\to\infty} n\,\mathrm{var}(\hat{a}_1) = c^{-2}\alpha^{-1}\;.$$

The excess of $\mathrm{var}(\hat{a}_2)$ over $\mathrm{var}(\hat{a}_1)$ can be split into parts:

$$c^{-2}\{\psi(\alpha) - \alpha^{-1}\} = \frac{1}{2} c^{-2}\alpha^{-1}(\alpha - \frac{1}{2})$$

and $\psi'(\theta_1) - \psi'(\theta_1+\theta_2)$.

The excess decreases as α increases and as c increases; the excess due to θ_1 and θ_2 is relatively less important when α and c are small. These features can be seen from the figures in Table 1.

Comparison of Tables 1 and 2 shows that the variables $\{X_2\}$ often provide unbiased estimators of c which are of comparable accuracy (e.g. with variances no more than twice as great) to $\hat{a}_1$. This is especially notable for the larger value of θ_1 and θ_2 - another feature which is indicated by (10) and (11). Except in 3 cases (out of 60) in Table 1, the estimator $\hat{a}_2$ (based on the arithmetic mean) has a smaller variance than $\hat{a}_3$ (based on the geometric mean). For the smaller values of c and α, the correlation between $\hat{a}_2$ and $\hat{a}_3$ is small enough to give some hope that the unbiased estimator

$$\hat{a}_4 = A\hat{a}_2 + (1-A)\hat{a}_3$$

with A chosen to minimize $\mathrm{var}(\hat{a}_4)$, suggested in [1], will give a useful reduction in variance. The last two columns of Table 1 give values of A and $\mathrm{var}(\hat{a}_4)$. The reduction in variance is certainly worthwhile for $c = 0.5$, but for $c = 1.0$ and 2.0 it does not seem to be of much importance. Note that as c increases the value of A becomes negative.

If θ_1 and θ_2 are large, with θ_1 large compared with θ_2, then X_{1j} and X_{2j} are highly correlated, and $\theta_1^{-1}(\theta_1+\theta_2)X_{2j}$ is a good unbiased estimator of X_{1j}. It would seem likely, therefore (in view of (3)) that

$$(12) \qquad \hat{a}_1' = (\theta_1+\theta_2)\,\Gamma(n\alpha)\,[\theta_1\,\Gamma(n\alpha+c^{-1})]^{-1}\left(\sum_{j=1}^{n} X_{2j}^{c}\right)^{1/c}$$

would be a good (though not unbiased) estimator of $\underline{a}$, in such cases. We note that the statistic

$$(13) \qquad \frac{\Gamma(\theta_1+\theta_2+c)\,\Gamma(\theta_1)}{\Gamma(\theta_1+c)\,\Gamma(\theta_1+\theta_2)} \cdot \frac{1}{n\,\alpha} \sum_{j=1}^{n} X_{2j}^{c}$$

is an unbiased estimator of a^c with (coefficient of variation)^2 equal to

$$(14) \qquad n^{-1}\left\{\frac{\alpha+1}{\alpha}\cdot\frac{\Gamma(\theta_1+2c)\Gamma(\theta_1)}{[\Gamma(\theta_1+c)]^2}\cdot\frac{[\Gamma(\theta_1+\theta_2+c)]^2}{\Gamma(\theta_1+\theta_2+2c)\Gamma(\theta_1+\theta_2)} - 1\right\}.$$

For the minimum variance unbiased estimator of a^c,

$$(n\alpha)^{-1}\sum_{j=1}^{n} X_{1j}^{c}$$

the (coefficient of variation)^2 is $(n\alpha)^{-1}$. On comparison with (14), we see that when $c = 1$, the efficiency of (13), as an estimator of a^c, is

$$\left[1 + \frac{\theta_2(1+\alpha)}{\theta_1(\theta_1+\theta_2+1)}\right]^{-1}.$$

Table 1 is based on tables to 5 significant figures for

$c = 0.5, 1.0, 2.0;\quad \alpha = 0.5, 1.0, 2.0, 3.0$

$\theta_1 , \theta_2 = 0.5, 1.0, 2.0, 5.0;\quad n = 10, 20, 50, 100, \infty$

calculated with an APL program devised by Mr. J. O. Kitchen, to whom we express our gratitude. Thanks are also due to Mrs. G. Ballard for assistance with the preparation of this paper.

REFERENCE

1. Mihram, G. A. and Hultquist, R. A. (1967) A bivariate warning-time/ failure time distribution, J. Amer. Statist. Assoc., vol. 62, pp. 589 - 599.

Table 1: COMPARISON OF ESTIMATORS OF a

c	α	θ_1	θ_2	n	$a^{-2}n\ \mathrm{var}(\hat{a}_2)$	$a^{-2}n\ \mathrm{var}(\hat{a}_3)$	$\mathrm{corr}(\hat{a}_2,\hat{a}_3)$	A	$a^{-2}n\ \mathrm{var}(\hat{a}_4)$
0.5	0.5	0.5	0.5	10	30.3		0.33	0.29	13.9
				20	27.7		0.32	0.32	13.4
				50	25.2	16.5	0.32	0.35	13.1
				100	24.2		0.32	0.36	12.9
				∞	23.0				
		0.5	1.0	100	25.0	20.0	0.30	0.42	14.4
		1.0	0.5		21.1	13.0	0.35	0.32	10.7
		1.0	1.0		21.5	14.6	0.33	0.36	11.5
		1.0	2.0		21.8	16.5	0.32	0.40	12.3
		2.0	1.0		20.6	12.1	0.35	0.30	10.1
		2.0	2.0		20.7	13.0	0.34	0.33	10.6
		1.0	5.0		22.0	19.0	0.31	0.45	13.3
		5.0	1.0		20.3	11.0	0.36	0.27	9.5
		5.0	5.0	10	22.6		0.37	0.26	10.3
				20	22.0		0.36	0.27	10.1
				50	20.9	11.7	0.35	0.29	10.0
				100	20.4		0.35	0.29	9.9
				∞	19.9				
0.5	2.0	0.5	0.5	10	5.7		0.59	0.29	3.7
				20	5.8		0.57	0.29	3.6
				50	5.9	4.0	0.56	0.29	3.6
				100	5.9		0.55	0.29	3.6
				∞					
		0.5	1.0	100	6.6	5.0	0.53	0.36	4.3
		1.0	0.5		3.3	3.0	0.64	0.44	2.6
		1.0	1.0		3.6	3.4	0.62	0.47	2.8
		1.0	2.0		3.8	4.0	0.60	0.52	3.1
		2.0	1.0		2.8	2.75	0.66	0.48	2.3
		2.0	2.0		2.9	3.0	0.65	0.51	2.5
		1.0	5.0		4.1	4.7	0.58	0.59	3.4
		5.0	1.0		2.6	2.4	0.68	0.44	2.1
		5.0	5.0	10	2.7		0.69	0.47	2.3
				20	2.7		0.68	0.48	2.2
				50	2.7	2.6	0.67	0.48	2.2
				100	2.7		0.67	0.48	2.2
				∞					

Table 1 (continued): Page 2

c	α	θ_1	θ_2	n	$a^{-2}n\ var(\hat{a}_2)$	$a^{-2}n\ var(\hat{a}_3)$	$corr(\hat{a}_2, a_3)$	A	$a^{-2}n\ var(\hat{a}_4)$
1.0	0.5	0.5	0.5	10	8.07		0.58	0.08	3.47
				20	8.23		0.57	0.08	3.46
				50	8.26	3.50	0.56	0.08	3.46
				100	8.25		0.56	0.09	3.46
				∞					
		0.5	1.0	100	9.00	4.40	0.53	0.16	4.24
		1.0	0.5		5.62	2.60	0.61	0.07	2.58
		1.0	1.0		5.92	3.00	0.59	0.13	2.94
		1.0	2.0		6.18	3.50	0.58	0.19	3.34
		2.0	1.0		5.15	2.375	0.62	0.06	2.36
		2.0	2.0		5.26	2.60	0.61	0.10	2.56
		1.0	5.0		6.41	4.14	0.55	0.27	3.80
		5.0	1.0		4.93	2.09	0.63	0.18	2.08
		5.0	5.0	10	4.61		0.65	0.06	2.26
				20	4.83		0.64	0.06	2.26
				50	4.97	2.27	0.63	0.06	2.26
				100	5.01		0.62	0.06	2.26
				∞					
1.0	2.0	0.5	0.5	10	3.29		0.72	-0.13	1.22
				20	3.60		0.70	-0.12	1.22
				50	3.80	1.25	0.68	-0.12	1.22
				100	3.86		0.68	-0.12	1.22
				∞					
		0.5	1.0	100	4.59	1.70	0.66	-0.05	1.69
		1.0	0.5		1.34	0.80	0.90	-0.07	0.30
		1.0	1.0		1.63	1.00	0.78	0.00	1.00
		1.0	2.0		1.89	1.25	0.76	0.10	1.24
		2.0	1.0		0.89	0.69	0.85	0.08	0.69
		2.0	2.0		1.00	0.80	0.84	0.16	0.79
		1.0	5.0		2.10	1.57	0.73	0.23	1.52
		5.0	1.0		0.68	0.54	0.87	0.06	0.54
		5.0	5.0	10	0.75		0.88	0.19	0.63
				20	0.75		0.87	0.18	0.63
				50	0.76	0.64	0.87	0.18	0.63
				100	0.76		0.86	0.18	0.63
				∞					

Table 1 (continued): Page 3

c	α	θ_1	θ_2	n	$a^{-2}n\ var(\hat{a}_2)$	$a^{-2}n\ var(\hat{a}_3)$	$corr(\hat{a}_2,\hat{a}_3)$	A	$a^{-2}n\ var(\hat{a}_4)$
2.0	0.5	0.5	0.5	10	3.90		0.72	-0.15	1.31
				20	4.21		0.70	-0.14	1.31
				50	4.40	1.36	0.69	-0.14	1.31
				100	4.46		0.69	-0.14	1.31
				∞					
		0.5	1.0	100	5.19	1.83	0.66	-0.07	1.81
		1.0	0.5		1.93	0.88	0.79	-0.18	0.86
		1.0	1.0		2.22	1.09	0.77	-0.11	1.08
		1.0	2.0		2.47	1.36	0.74	-0.004	1.36
		2.0	1.0		1.47	0.77	0.81	-0.18	0.75
		2.0	2.0		1.58	0.88	0.80	-0.10	0.88
		1.0	5.0		2.69	1.69	0.72	+0.12	1.67
		5.0	1.0		1.26	0.62	0.82	-0.26	0.59
		5.0	5.0	10	1.24		0.83	-0.18	0.70
				20	1.29		0.82	-0.17	0.70
				50	1.33	0.71	0.81	-0.17	0.70
				100	1.34		0.81	-0.17	0.70
				∞					
2.0	2.0	0.5	0.5	10	2.71		0.77	-0.23	0.59
				20	3.05		0.75	-0.26	0.59
				50	3.28	0.70	0.74	-0.25	0.59
				100	3.37		0.74	-0.24	0.59
				∞					
		0.5	1.0	100	4.09	1.04	0.71	-0.20	0.95
		1.0	0.5		0.86	0.36	0.85	-0.41	0.31
		1.0	1.0		1.15	0.51	0.84	-0.34	0.47
		1.0	2.0		1.40	0.70	0.82	-0.22	0.67
		2.0	1.0		0.41	0.27	0.92	-0.48	0.26
		2.0	2.0		0.52	0.38	0.90	-0.32	0.35
		1.0	5.0		1.61	0.94	0.79	-0.54	0.94
		5.0	1.0		2.01	1.64	0.96	-0.52	1.59
		5.0	5.0	10	2.72		0.95	-0.18	2.34
				20	2.75		0.94	-0.18	2.34
				50	2.76	2.35	0.94	-0.18	2.34
				100	2.77		0.94	-0.17	2.34
				∞					

Table 2: MINIMUM VARIANCE OF UNBIASED ESTIMATOR OF a

$(V = a^{-2} n \, \mathrm{var}(\hat{a}_1))$

c =	0.5				1.0				2.0			
n	α	V	α	V	α	V	α	V	α	V	α	V
10		8.67		2.05		2.00		0.50		0.51		0.126
20		8.36		2.02		2.00		0.50		0.50		0.125
50	0.5	8.15	2.0	2.01	0.5	2.00	2.0	0.50	0.5	0.50	2.0	0.125
100		8.08		2.00		2.00		0.50		0.50		0.125
∞		8.00		2.00		2.00		0.50		0.50		0.125